Semester-03 Diff. eq.

5. Linear Equations with constant coefficient

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5.1 The general linear diff. eqn of nth order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = X \quad \text{--- (1)}$$

where X and the coefficients $P_1, P_2, P_3, \dots, P_n$ are functions of x only.

5.2 Linear Equations with Constant Coefficients :-

The general linear equation with constant coefficient

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = X \quad \text{--- (2)}$$

Where $P_1, P_2, \dots, P_n$ are Constants and X is a function of x or Const.

The solution of (2) consists two parts:

(i) The complementary function (C.F.)

(ii) The particular Integral (P.I.)

Working rule:- (i) Mark right hand member viz $X=0$ and find the complete solution of the reduced equation. This will give us complementary Function.

(ii) Find by an means ~~of~~ the particular solution of the equation itself. This will give us Particular Integral.

(iii) The general solution $= C.F. + P.I.$

5.3 Symbolic Representation :

D stands for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$ ... and D^n for $\frac{d^n}{dx^n}$

Eq (2) becomes,

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_n) y = 0, \Rightarrow \{f(D)\} y = 0$$

5.4 Formation of Auxiliary Equation :-

The auxiliary equation of eq (2) is

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_n) = 0 \quad \{X = 0\}$$

or, $f(D) = 0 \Rightarrow D = ?$

or, $f(m) = 0 \Rightarrow m = ?$ {Replace D by m}

5.5 Method of finding Complementary Function:

The method of finding C.F. of ~~a given equation~~
This will depend on the nature of roots of the auxiliary equation according as

(i) The roots are real and unequal
(ii) The roots are real and equal
(iii) The roots are conjugate complex and not repeated
(iv) The roots are conjugate complex and Repeated

Case I Roots are real and unequal:-

If $m_1, m_2, m_3, \ldots, m_n$ be the different roots of the auxiliary equation $f(m)=0$. Then the Complementary function

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$$

where $C_1, C_2, C_3, \ldots, C_n$ are arbitrary Constants.

Case II Roots are real and equal:-

(i) If m_1 be repeated root twice of $f(m)=0$

Then, $C.F = (C_1 + C_2 x) e^{m_1 x}$

(ii) If m_2 be repeated root thrice of $f(m)=0$

Then, $C.F. = (C_1 + C_2 x + C_3 x^2) e^{m_1 x}$

etc

Case III Roots Conjugate Complex and Non-Repeated

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ be the roots of auxiliary equation $f(m)=0$. Then,

$$C.F. = e^{\alpha x} \cdot (C_1 \cos\beta x + C_2 \sin\beta x)$$

~~Where A = C1 + C2 and~~

Case IV Roots Conjugate and Repeated

When the Conjugate roots $\alpha \pm i\beta$ are each repeated r times,

$$C.F. = e^{\alpha x} \cdot \left[(A_1 + A_2 x + \cdots + A_r x^{r-1}) \cos\beta x + (B_1 + B_2 x + \cdots + B_r x^{r-1}) \sin\beta x\right]$$

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Ex ① Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Soln:- The auxiliary ~~Eq~~ Equation, $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2 \text{ or } 3$

The general Solution, $y = C_1 \cdot e^{2x} + C_2 \cdot e^{3x}$ Ans

Ex ② Solve: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

The auxiliary Equation, $D^2 - D - 2 = 0$

$\Rightarrow D^2 - 2D + D - 2 = 0$

$\Rightarrow D(D-2) + 1(D-2) = 0$

$\Rightarrow (D-2)(D+1) = 0$

$\therefore D = 2, -1$

The general Soln: $y = C_1 e^{2x} + C_2 e^{-x}$
(C.F.)

Ex ③ Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

Soln:- The auxiliary eqn: ~~D^2~~ $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$

The general Soln: $y = (C_1 + C_2 x)e^{3x}$ Ans
(C.F.)

Ex ④ Solve: $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Soln:- The auxiliary eqn: $m^3 - 3m + 2 = 0$

$\Rightarrow m^3 - m - 2m + 2 = 0$

$\Rightarrow m(m^2 - 1) - 2(m-1) = 0$

$\Rightarrow (m-1)[m(m+1) - 2] = 0$

$\Rightarrow (m-1)[m^2 + m - 2] = 0$

$\Rightarrow (m-1)[m^2 + 2m - m - 2] = 0$

$\Rightarrow (m-1)[m(m+2) - 1(m+2)] = 0$

$\Rightarrow (m-1)(m-1)(m+2) = 0 \Rightarrow m = 1, 1, -2$

The general Solution,

$y = (C_1 + C_2 x)e^{x} + C_3 e^{-2x}$ Ans

Ex ⑤ solve: $\frac{d^2y}{dx^2} + 4y = 0$.

Soln:- The auxiliary ~~Equation~~ Equation, $m^2+4=0$

$$\Rightarrow m^2-(2i)^2=0$$
$$\Rightarrow (m-2i)(m+2i)=0$$
$$\therefore m = 2i, -2i \quad \{\alpha=0, \beta=2\}$$

The general Solution: (C.F.)

$$y = e^{\alpha x}\cdot(C_1\cos\beta x + C_2\sin\beta x)$$
$$= e^{0\cdot x}\cdot(C_1\cos 2x + C_2\sin 2x)$$
$$\boxed{y = C_1\cos 2x + C_2\sin 2x}$$

Ex ⑥ Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Soln:- The auxiliary eqn, $m^2+m+1=0$

$$\therefore m = \frac{-1\pm\sqrt{1^2-4\times1\times1}}{2\times1} = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$$

$$m = \frac{-1}{2}+\frac{\sqrt{3}i}{2}, \frac{-1}{2}-\frac{\sqrt{3}i}{2}, \quad \alpha = -\frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$$

The general Soln: $y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

$$y = e^{-\frac{1}{2}x}\left(C_1\cos\frac{\sqrt{3}x}{2} + C_2\sin\frac{\sqrt{3}x}{2}\right)$$

Ex ⑦ solve: $\frac{d^3y}{dx^3} - 8y = 0$

Soln: The auxiliary eqn: $m^3-8=0 \Rightarrow m^3-2^3=0$

$$\Rightarrow (m-2)(m^2+2m+4)=0$$
$$\therefore m=2,\ m = \frac{-2\pm\sqrt{2^2-4\times1\times4}}{2\times1} = \frac{-2\pm2\sqrt{3}i}{2}$$
$$m=2,\ m = -1+\sqrt{3}, \text{ or } -1-\sqrt{3}i$$

The general soln:

$$y = C_1e^{2x} + e^{(-1)\cdot x}\{C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x\}$$

Ans

~~Ex ⑦ solve:~~